Differential Geometry III: Riemannian geometry

5.1 (a) Let $X \in \Gamma(\mathcal{M})$ be a *Killing vector field* on the Riemannian manifold $(\mathcal{M}, g)$. Show that, for any $V, W \in \Gamma(\mathcal{M})$,

$$g\big(\nabla_V X, W\big) + g\big(\nabla_W X, V\big) = 0,$$

where ∇ is the Levi-Civita connection of g. Deduce that, if $\gamma : (a, b) \to \mathcal{M}$ is a geodesic of $(\mathcal{M}, g)$, then the function $t \to g(X|_{\gamma(t)}, \dot{\gamma})$ is constant for $t \in (a, b)$

Remark. Any function $F : T\mathcal{M} \to \mathbb{R}$ such that $F(\gamma(t), \dot{\gamma}(t))$ is constant when γ is a geodesic is called a *constant of motion* for the geodesic flow.

(b) Let $\zeta : (-1, 1) \to \mathbb{R}^2$, $\zeta(u) = \big(x(u), y(u)\big)$ be a smooth curve parametrized with *unit speed* and contained in the upper half plane, i.e. $y(u) > 0$ for all $u \in (-1, 1)$. Let S be the *surface of revolution* in $\mathbb{R}^3$ obtained by rotating the curve ζ around the x-axis, i.e. S is parametrized by the map $\Psi : (-1, 1) \times [0, 2\pi)$,

$$\Psi(u, \varphi) = \big(x(u), y(u)\cos(\varphi), y(u)\sin(\varphi)\big).$$

Let also g be the metric induced on S by the Euclidean metric g_E in $\mathbb{R}^3$. Show that the vector field $\Phi = \frac{\partial}{\partial\varphi}$ is a Killing vector field on (S, g). Find a closed formula for any geodesic $\gamma : (-T, T) \to (S, g)$, $t \to \big(u(t), \varphi(t)\big)$ *(Hint: Use the fact that $g(\dot{\gamma}, \Phi)$ and $g(\dot{\gamma}, \dot{\gamma})$ are conserved along γ to obtain a simple expression for $\dot{\gamma} = (\frac{du}{dt}, \frac{d\varphi}{dt})$).*

Solution. (a) Recall that a Killing vector field X of $(\mathcal{M}, g)$ satisfies the relation

$$\mathcal{L}_X g = 0$$

(see Exercise 4.3). Therefore, for any $V, W \in \Gamma(\mathcal{M})$, we can calculate using the fact that $\mathcal{L}_X$ commutes with contractions and satisfies the product rule with respect to tensor products of tensors:

$$X\big(g(V, W)\big) = \mathcal{L}_X\big(g(V, W)\big) = (\mathcal{L}_X g)(V, W) + g\big(\mathcal{L}_X V, W\big) + g\big(V, \mathcal{L}_X W\big) = 0 + g\big([X, V], W\big) + g\big(V, [X, W]\big).$$

On the other hand, we can express the left hand side of the above relation using the covariant derivative ∇_X with respect to the Levi-Civita connection of g as follows:

$$X\big(g(V, W)\big) = g\big(\nabla_X V, W\big) + g\big(V, \nabla_X W\big).$$

Since the left hand sides of the above two relations are the same, we obtain:

$$\begin{aligned} g\big(\nabla_X V, W\big) + g\big(V, \nabla_X W\big) &= g\big([X, V], W\big) + g\big(V, [X, W]\big) \\ \Rightarrow \quad g\big(\nabla_X V - [X, V], W\big) &+ g\big(V, \nabla_X W - [X, W]\big) = 0. \end{aligned} \tag{1}$$

Using the fact that the Levi-Civita connection is torsion-free, i.e.

$$\nabla_X Y = \nabla_Y X + [X, Y] \quad \text{for any } X, Y \in \Gamma(\mathcal{M}), \tag{2}$$

we obtain from (1) using (2) for $Y = V, W$:

$$g\big(\nabla_V X, W\big) + g\big(V, \nabla_W X\big) = 0 \tag{3}$$

Let $\gamma : (a,b) \to \mathcal{M}$ be a smooth curve. Using (3) for $V = W = \dot{\gamma}$, we infer

$$g\big(\nabla_{\dot{\gamma}} X, \dot{\gamma}\big) = 0.$$

Assuming, in addition, that γ is a geodesic, i.e.

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0,$$

we can readily calculate:

$$\frac{d}{dt} g\big(X|_{\gamma(t)}, \dot{\gamma}(t)\big) = \dot{\gamma}\big(g(X, \dot{\gamma})\big) = g\big(\nabla_{\dot{\gamma}} X, \dot{\gamma}\big) + g\big(X, \nabla_{\dot{\gamma}} \dot{\gamma}\big) = 0,$$

i.e. $g\big(X|_{\gamma(t)}, \dot{\gamma}(t)\big)$ is constant along for $t \in (a,b)$.

(b) Let us denote with $(\bar{x}, \bar{y}, \bar{z})$ the Cartesian coordinates on $\mathbb{R}^3$, so that

$$g_E = (d\bar{x})^2 + (d\bar{y})^2 + (d\bar{z})^2.$$

We can readily calculate that the pull-back metric $g = \Psi_* g_E$ in the (u, φ) coordinates on S takes the following form (noting that Ψ maps $(u,\varphi) \to (\bar{x}, \bar{y}, \bar{z}) = (x(u),\, y(u)\cos\varphi,\, y(u)\sin\varphi)$):

$$\begin{aligned}
g &= \Psi_* g_E \\
&= \Psi_*\big((d\bar{x})^2 + (d\bar{y})^2 + (d\bar{z})^2\big) \\
&= \big(d\,x(u)\big)^2 + \big(d(y(u)\cos\varphi)\big)^2 + \big(d(y(u)\sin\varphi)\big)^2 \\
&= (\dot{x}(u))^2 du^2 + (\dot{y}(u))^2 \cos^2\varphi du^2 + (y(u))^2 \sin^2\varphi d\varphi^2 + (\dot{y}(u))^2 \sin^2\varphi du^2 + (y(u))^2 \cos^2\varphi d\varphi^2 \\
&= \big((\dot{x}(u))^2 + (\dot{y}(u))^2\big) du^2 + (y(u))^2 d\varphi^2.
\end{aligned}$$

In view of our assumption that $\zeta(u) = (x(u), y(u))$ is parametrized with unit speed, we have

$$(\dot{x}(u))^2 + (\dot{y}(u))^2 = 1$$

and, therefore:

$$g = du^2 + (y(u))^2 d\varphi^2.$$

In the (u,φ) coordinate system on S, the vector field $\Phi = \frac{\partial}{\partial\varphi}$ is a coordinate vector field. Therefore, the flow map $\mathcal{F}_t : S \to S$ associated to Φ is simply the coordinate translation $(u,\varphi) \to (u, \varphi + t \mod 2\pi)$ (geometrically, this corresponds to a rotation of $S \subset \mathbb{R}^3$ around the x axis). Since the components of the metric g in the (u,φ) coordinates are *independent* of φ, we infer that, for each $t \in \mathbb{R}$, the flow map $\mathcal{F}_t$ is an isometry of (S,g) and, therefore, Φ is a Killing vector field for (S,g).

Remark. In general, if $(x^1, \ldots, x^n)$ is a local coordinate chart on a manifold $\mathcal{M}$ and $X = \frac{\partial}{\partial x^1}$, the flow map associated to X is simply the translation in the x^1 coordinate. Therefore, for any tensor field T on $\mathcal{M}$ of type (k,l), we have

$$\big(\mathcal{L}_X T\big)^{i_1\ldots i_k}{}_{j_1\ldots j_l} = \frac{\partial}{\partial x^1} T^{i_1\ldots i_k}{}_{j_1\ldots j_l}.$$

In particular, $L_X T = 0$ if and only if the components $T^{i_1\dots i_k}{}_{j_1\dots j_l}$ in the $(x^1, \dots, x^n)$ coordinate chart do not depend x^1.

In the example of this exercise, we could have also inferred that the flow of Φ is an isometry of (S, g) by noting that, when viewed as a vector field on $\mathbb{R}^3$, Φ is the generator of rotations for $(\mathbb{R}^3, g_E)$, which are isometries for g_E and leave the surface S invariant.

Let $\gamma : (-T, T) \to S$, $t \to \big(u(t), \varphi(t)\big)$ be a geodesic. We know that $g(\dot\gamma, \dot\gamma)$ is constant along γ; by reparametrizing γ, we can assume without loss of generality that $g(\dot\gamma, \dot\gamma) = 1$, i.e.

$$\dot u^2(t) + \big(y(u(t))\big)^2 \dot\varphi^2(t) = 1.$$

Moreover, since $\Phi = \frac{\partial}{\partial\varphi}$ is a Killing vector field for (S, g), by part (a) of this exercise we know that $g(\dot\gamma, \frac{\partial}{\partial\varphi})$ is also constant along γ, i.e. there exists some $\lambda \in \mathbb{R}$ such that

$$\big(y(u(t))\big)^2 \dot\varphi(t) = \lambda.$$

Combining the above two relations, we obtain:

$$\begin{aligned} \frac{du}{dt}(t) &= \pm\sqrt{1 - \frac{\lambda^2}{\big(y(u(t))\big)^2}}, \\ \frac{d\varphi}{dt}(t) &= \frac{\lambda}{\big(y(u(t))\big)^2}. \end{aligned}$$

In principle, the above system can be "explicitly" solved: If G_λ is a function such that

$$G'_\lambda(x) = \frac{1}{\sqrt{1 - \frac{\lambda^2}{(y(x))^2}}},$$

then

$$\begin{aligned} G_\lambda\big(u(t)\big) &= G_\lambda\big(u(0)\big) \pm t, \\ \varphi(t) &= \varphi(0) + \int_0^t \frac{\lambda}{\big(y(u(s))\big)^2}\, ds \mod 2\pi. \end{aligned}$$

5.2 The *Poincaré half-plane* is the domain $\mathbb{H}^2 = \big\{(x, y) \in \mathbb{R}^2 : y > 0\big\}$ in $\mathbb{R}^2$ equipped with the Riemannian metric

$$g_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}.$$

(a) Setting $z = x + iy$, show that the map $f : \mathbb{H}^2 \to \mathbb{H}^2$ defined by

$$f(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$ is an *isometry* for $g_{\mathbb{H}}$.

(b) Show that the geodesic equation for $g_{\mathbb{H}}$ takes the form

$$\ddot{x} = \frac{2\dot{x}\dot{y}}{y}, \quad \ddot{y} = \frac{\dot{y}^2 - \dot{x}^2}{y}.$$

(c) Show that $\frac{\dot{x}^2+\dot{y}^2}{y^2}$ and $\frac{\dot{x}}{y^2}$ are constant along a geodesic (i.e. are constants of motion for the geodesic flow). Is the conserved quantity $\frac{\dot{x}}{y^2}$ a constant of motion associated to a Killing vector field of $(\mathbb{H}^2, g_{\mathbb{H}})$, in the spirit of Exercise 5.1? What is the shape of a geodesic curve in $(\mathbb{H}^2, g_{\mathbb{H}})$?

Remark. The *Poincaré half-plane* is a model for the *hyperbolic plane.*

Solution. (a) It suffices to check that the statement is true in the following two cases:

1. $c = 0$ and $d = 1$,
2. $a = 1$, $b = d = 0$, $c = -1$.

The general map f can then be written as a sequence of (at most three) compositions of special maps falling in the categories 1 or 2 above; since the composition of isometries is again an isometry, the general statement would follow.

- *The case $f(z) = az+b$, $a > 0$:* It can be readily verified that, in this case, the map $f : \mathbb{H}^2 \to \mathbb{H}^2$ is a bijection which is also bi-continuous. Thus, in order to prove that it is an isometry, we only have to check that
$$g_{\mathbb{H}} = f_* g_{\mathbb{H}}.$$
Since $f : \mathbb{H}^2 \to \mathbb{H}^2$ maps (x, y) to $(\bar{x}, \bar{y}) = \big(\mathrm{Re}(f(x+iy)), \mathrm{Im}(f(x+iy))\big) = (ax+b, ay)$, we can readily compute
$$\begin{aligned} f_* g_{\mathbb{H}} &= f_*\Big(\frac{d\bar{x}^2 + d\bar{y}^2}{\bar{y}^2}\Big) \\ &= \frac{\big(d(ax+b)\big)^2 + \big(d(ay)\big)^2}{(ay)^2} \\ &= \frac{a^2dx^2 + a^2dy^2}{a^2y^2} \\ &= \frac{dx^2 + dy^2}{y^2} \\ &= g_{\mathbb{H}}. \end{aligned}$$

- *The case $f(z) = -\frac{1}{z}$:* In this case it can be also readily verified that the map $f : \mathbb{H}^2 \to \mathbb{H}^2$ is a bijection which is also bi-continuous. Since $f : (x, y) \to (\bar{x}, \bar{y}) = \big(\mathrm{Re}(f(x+iy)), \mathrm{Im}(f(x+iy))\big) = \big(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\big)$, we can readily calculate

$$f_* g_{\mathbb{H}} = f_*\Big(\frac{d\bar{x}^2 + d\bar{y}^2}{\bar{y}^2}\Big)$$

$$= \frac{\Big(d\big(\frac{-x}{x^2+y^2}\big)\Big)^2 + \Big(d\big(\frac{y}{x^2+y^2}\big)\Big)^2}{\Big(\frac{y}{x^2+y^2}\Big)^2}$$

$$= \frac{\Big(\frac{-dx}{x^2+y^2} - xd\big(\frac{1}{x^2+y^2}\big)\Big)^2 + \Big(\frac{dy}{x^2+y^2} + yd\big(\frac{1}{x^2+y^2}\big)\Big)^2}{\frac{y^2}{(x^2+y^2)^2}}$$

$$= \frac{\frac{dx^2}{(x^2+y^2)^2} + 2xdx\frac{1}{x^2+y^2}d\big(\frac{1}{x^2+y^2}\big) + x^2\Big(d\big(\frac{1}{x^2+y^2}\big)\Big)^2 + \frac{dy^2}{(x^2+y^2)^2} + 2ydy\frac{1}{x^2+y^2}d\big(\frac{1}{x^2+y^2}\big) + y^2\Big(d\big(\frac{1}{x^2+y^2}\big)\Big)^2}{\frac{y^2}{(x^2+y^2)^2}}$$

$$= \frac{1}{y^2}\left(dx^2 + dy^2 + (x^2+y^2)(2xdx + 2ydy)d\big(\frac{1}{x^2+y^2}\big) + (x^2+y^2)^3\Big(d\big(\frac{1}{x^2+y^2}\big)\Big)^2\right)$$

$$= \frac{1}{y^2}\left(dx^2 + dy^2 + (x^2+y^2)d(x^2+y^2)\big[-\frac{d(x^2+y^2)}{(x^2+y^2)^2}\big] + (x^2+y^2)^3\Big(-\frac{d(x^2+y^2)}{(x^2+y^2)^2}\Big)^2\right)$$

$$= \frac{1}{y^2}\left(dx^2 + dy^2 - (x^2+y^2)^{-1}\big(d(x^2+y^2)\big)^2 + (x^2+y^2)^{-1}\big(d(x^2+y^2)\big)^2\right)$$

$$= \frac{1}{y^2}\big(dx^2+dy^2\big)$$

$$= g_{\mathbb{H}}.$$

(b) Let us start by computing the Christoffel symbols for $g_{\mathbb{H}}$: Using the index 1 for the x variable and 2 for the y variable, we have

$$g_{11} = g_{22} = y^{-2}, \quad g_{12} = 0 \quad \text{and} \quad g^{11} = g^{22} = y^2, \quad g^{12} = 0$$

and, using the formula $\Gamma^k_{ij} = \frac{1}{2}g^{kl}\big(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\big)$:

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = 0, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\Gamma^2_{11} = -y^{-1}.$$

Therefore, if $t \to \big(x(t), y(t)\big)$ is a geodesic, the geodesic equation $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$ takes the form

$$\ddot{x} - \frac{\dot{x}\dot{y}}{y} = 0, \tag{4}$$

$$\ddot{y} - \frac{\dot{y}^2 - \dot{x}^2}{y} = 0. \tag{5}$$

(c) Using the geodesic equations (4)–(5), we can easily verify that, if $t \to \gamma(t) = \big(x(t), y(t)\big)$ is a geodesic for $(\mathbb{H}^2, g_{\mathbb{H}})$ and

$$E = \frac{\dot{x}^2 + \dot{y}^2}{y^2},$$

$$P = \frac{\dot{x}}{y^2},$$

then

$$\frac{d}{dt}E = \frac{d}{dt}P = 0.$$

Note that $E = g_{\mathbb{H}}(\dot{\gamma}, \dot{\gamma})$ and $P = g_{\mathbb{H}}(\dot{\gamma}, \frac{\partial}{\partial x})$. In particular, in the language of Ex. 5.1, P is the constant of motion associated to the Killing vector field $\frac{\partial}{\partial x}$ of $(\mathbb{H}^2, g_{\mathbb{H}})$ (note that it is straightforward to verify that $\frac{\partial}{\partial x}$ is a Killing vector field, since the components of $g_{\mathbb{H}}$ in the (x, y) coordinates do not depend on x).

In order to determine the shape of a geodesic γ, we will have to distinguish two cases, depending on the value of the conserved quantity P (note that $E > 0$ unless γ is the constant curve):

- If $P = 0$, then the expression for P implies that $\dot{x} = 0$, i.e. γ (when maximally extended) is the half line $\{x = \text{const}\} \cap \{y > 0\}$.

- If $P \neq 0$, then we can solve for $\dot{x}$ and $\dot{y}$ from the expressions for E, P as follows:

$$\dot{x} = y^2 P,$$
$$\dot{y} = \pm y\sqrt{E - y^2 P^2}.$$

Since $P \neq 0$ and, therefore, $\dot{x} \neq 0$, we can use x as a parametrization of our curve γ, i.e. think of γ as a curve $x \to (x, y(x))$. In this case, we can calculate

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \pm\frac{\sqrt{\frac{E}{P^2} - y^2}}{y} \quad \Leftrightarrow \quad \frac{d}{dx}\Big(\sqrt{\frac{E}{P^2} - y^2}\Big) = \pm 1.$$

Notice that this is the ODE satisfied by the graph of a circle: If we integrate both sides, we obtain that

$$\sqrt{\frac{E}{P^2} - y^2} = \pm(x - x_*)$$

for some constant $x_* \in \mathbb{R}$. In particular, in this case, $\gamma(t)$ moves along a half-circle of radius $\frac{\sqrt{E}}{|P|}$ which intersects $y = 0$ orthogonally.

5.3 Let $(\mathcal{M}^n, g)$ be a smooth Riemannian manifold and let $\gamma : [0, 1] \to \mathcal{M}$ be a geodesic.

(a) Show that there exist a set of vector fields $\{E_i\}_{i=1}^n$ defined along the curve γ satisfying all of the following conditions:

- For any $t \in [0, 1]$, the tangent vectors $\{E_i|_{\gamma(t)}\}_{i=1}^n$ at $\gamma(t)$ form an orthonormal basis of $T_{\gamma(t)}\mathcal{M}$.
- $E_1|_{\gamma(t)}$ is parallel to $\dot{\gamma}(t)$.
- The vector fields E_i are parallel translated along γ, i.e. $\nabla_{\dot{\gamma}} E_i = 0$, $i = 1, \ldots, n$.

(b) Show that, if $\{E_i\}_{i=1}^n$ is a set of vector fields along γ as above and X is any other vector field along γ, then X is parallel-translated along γ if and only if the components of $X|_{\gamma(t)}$ in the basis $\{E_i|_{\gamma(t)}\}_{i=1}^n$ of $T_{\gamma(t)}\mathcal{M}$ are constant as functions of t.

Solution. (a) Let us pick a basis $\{\xi_i\}_{i=1}^n$ for the vector space $T_{\gamma(0)}\mathcal{M}$ which satisfies the following properties:

- $\xi_1 = \frac{\dot\gamma(0)}{\|\dot\gamma(0)\|}$,
- $\{\xi_i\}_{i=1}^n$ is orthonormal with respect to $g|_{\gamma(0)}$.

Note that such a basis can always be chosen (in a non-unique way) using the Gram-Schmidt process starting from any (not necessarily orthonormal) basis $\{\tilde\xi_i\}_{i=1}^n$ for which $\tilde\xi_1 = \dot\gamma(0)$.

For any $i = 1,\dots,n$, let $E_i \in \Gamma_\gamma$ be the the parallel translation of ξ_i along γ, i.e. the *unique* vector field along γ which satisfies

$$\nabla_{\dot\gamma} E_i = 0 \quad \text{and} \quad E_i|_{t=0} = \xi_i.$$

We will show that $\{E_i\}_{i=1}^n$ satisfies the required conditions:

- For any $i, j \in \{1,\dots,n\}$, we have

  $$\frac{d}{dt}\big(g(E_i,E_j)|_{\gamma(t)}\big) = \dot\gamma\big(g(E_i,E_j)\big) = g\big(\nabla_{\dot\gamma}E_i, E_j\big) + g\big(E_i, \nabla_{\dot\gamma}E_i\big) = 0$$

  since $\nabla_{\dot\gamma}E_i = \nabla_{\dot\gamma}E_j = 0$. Therefore,

  $$g(E_i,E_j)|_{\gamma(t)} = g(E_i,E_j)|_{\gamma(0)} = g(\xi_i,\xi_j) = \delta_{ij},$$

  i.e. $\{E_i\}_{i=1}^n$ is orthonormal.

- Since γ is a geodesic, $\nabla_{\dot\gamma}\dot\gamma = 0$; therefore, $\|\dot\gamma(t)\|$ is constant in t and the vector field $T = \frac{\dot\gamma(t)}{\|\dot\gamma(t)\|}$ satisfies

  $$\nabla_{\dot\gamma}T = 0 \quad \text{and} \quad T|_{t=0} = \xi_1.$$

  By the uniqueness of parallel transport, $E_1 = T$, i.e. $E_1 \parallel \dot\gamma$.

(b) Let $\{E_i\}_{i=1}^n$ be a parallel-translated orthonormal basis of the tangent space of $\mathcal{M}$ along γ as in part (a). For any $X \in \Gamma_\gamma$, the components X^i of X with respect to the basis $\{E_i\}_{i=1}^n$ are functions $X^i : [0,1] \to \mathbb{R}$ so that

$$X|_{\gamma(t)} = X^i(t)E_i|_{\gamma(t)}.$$

Therefore, we compute using the product rule for ∇:

$$\begin{aligned}
\nabla_{\dot\gamma}X &= \nabla_{\dot\gamma}\big(X^i(t)E_i|_{\gamma(t)}\big)\\
&= \frac{dX^i}{dt}E_i + X^i\nabla_{\dot\gamma}E_i\\
&= \frac{dX^i}{dt}E_i
\end{aligned}$$

since $\nabla_{\dot\gamma}E_i = 0$. Therefore, $\nabla_{\dot\gamma}X = 0$ if and only if $\frac{dX^i}{dt} = 0$ for $i = 1,\dots,n$.

5.4 Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $\Psi : \Omega \to \mathbb{R}^N$ ($N > n$) be a smooth immersion; let also g be the Riemannian metric induced on Ω by the Euclidean metric g_E on $\mathbb{R}^N$. Recall that, for for any point $p \in \Omega$ and any local coordinate system $(x^1, \dots, x^n)$ around p, the tangent space $T_{\Psi(p)}\Psi(\Omega)$ of the sumbanifold $\Psi(\Omega) \subset \mathbb{R}^N$ (i.e. the image of the map $d\Psi_p : T_p\Omega \to T_{\Psi(p)}\mathbb{R}^N$) is spanned by the vectors $\{\partial_i \Psi\}_{i=1}^n$. Let us denote with $\Pi^\top_{\Psi(p)} : T_{\Psi(p)}\mathbb{R}^N \to T_{\Psi(p)}\Psi(\Omega)$ the *orthogonal projection* with respect to the Euclidean inner product on $T_{\Psi(p)}\mathbb{R}^N$. Show that the Christoffel symbols of the Levi-Civita connection for g in the *Cartesian* coordinate system $(x^1, \dots, x^n)$ on $\Omega \subset \mathbb{R}^n$ satisfy for any $p \in \Omega$:

$$\Pi^\top_{\Psi(p)}\Big(\frac{\partial^2 \Psi}{\partial x^i \partial x^j}(p)\Big) = \Gamma^k_{ij}(p)\partial_k \Psi(p).$$

Solution. Let $(x^1, \dots, x^n)$ be the Cartesian coordinate system on $\Omega \subset \mathbb{R}^n$ and $(y^1, \dots, y^N)$ the Cartesian coordinate system on $\mathbb{R}^N$. The induced metric g on Ω by the immersion Ψ takes the following form:

$$g_{ij} = g_E\big(\Psi^*(\frac{\partial}{\partial x^i}), \Psi^*(\frac{\partial}{\partial x^j})\big) = \delta_{AB}\frac{\partial \Psi^A}{\partial x^i}\frac{\partial \Psi^B}{\partial x^j} = \big\langle \frac{\partial \Psi}{\partial x^i}, \frac{\partial \Psi}{\partial x^j}\big\rangle_{g_E}.$$

Therefore, we compute that the Christoffel symbols of the Levi-Civita connection of g take the following form:

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}g^{kl}\big(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\big)\\
&= \frac{1}{2}g^{kl}\big(\partial_i(\delta_{AB}\frac{\partial \Psi^A}{\partial x^l}\frac{\partial \Psi^B}{\partial x^j}) + \partial_j(\delta_{AB}\frac{\partial \Psi^A}{\partial x^l}\frac{\partial \Psi^B}{\partial x^i}) - \partial_l(\delta_{AB}\frac{\partial \Psi^A}{\partial x^i}\frac{\partial \Psi^B}{\partial x^j})\big)\\
&= \frac{1}{2}g^{kl}\delta_{AB}\Big(\frac{\partial^2 \Psi^A}{\partial x^i \partial x^l}\cdot\frac{\partial \Psi^B}{\partial x^j} + \frac{\partial \Psi^A}{\partial x^l}\cdot\frac{\partial^2 \Psi^B}{\partial x^i \partial x^j} + \frac{\partial^2 \Psi^A}{\partial x^j \partial x^l}\cdot\frac{\partial \Psi^B}{\partial x^i} + \frac{\partial \Psi^A}{\partial x^l}\cdot\frac{\partial^2 \Psi^B}{\partial x^i \partial x^j}\\
&\qquad - \frac{\partial^2 \Psi^A}{\partial x^i \partial x^l}\cdot\frac{\partial \Psi^B}{\partial x^j} - \frac{\partial \Psi^A}{\partial x^i}\cdot\frac{\partial^2 \Psi^B}{\partial x^j \partial x^l}\Big)\\
&= g^{kl}\big\langle \frac{\partial \Psi}{\partial x^l}, \frac{\partial^2 \Psi}{\partial x^i \partial x^j}\big\rangle_{g_E}
\end{aligned}$$

(where, in passing to the last line in the calculation above, we used the fact that $\delta_{AB} = \delta_{BA}$ to show that $\delta_{AB}\frac{\partial^2 \Psi^A}{\partial x^j \partial x^l}\cdot\frac{\partial \Psi^B}{\partial x^i} = \delta_{AB}\frac{\partial^2 \Psi^B}{\partial x^j \partial x^l}\cdot\frac{\partial \Psi^A}{\partial x^i}$, i.e. the blue terms cancel out). Therefore,

$$\Gamma^k_{ij}\frac{\partial \Psi}{\partial x^k} = \big\langle \frac{\partial \Psi}{\partial x^l}, \frac{\partial^2 \Psi}{\partial x^i \partial x^j}\big\rangle_{g_E} g^{kl}\frac{\partial \Psi}{\partial x^k}.$$

In order to complete the proof of the exercise, we therefore have to show that the projection operator $\Pi^\top_{\Psi(p)}$ takes the following form for any $Z \in T_{\Psi(p)}\mathbb{R}^N$:

$$\Pi^\top_{\Psi(p)}(Z) = \big\langle \frac{\partial \Psi}{\partial x^l}(p), Z\big\rangle_{g_E} g^{kl}|_p \frac{\partial \Psi}{\partial x^k}(p).$$

To this end, we simply have to verify that the right hand side in the relation above vanishes if $Z \perp T_{\Psi(p)}\Psi(\Omega)$ and is equal to Z if $Z \in T_{\Psi(p)}\Psi(\Omega)$ (since this is the definition of the projection operator $\Pi^\top_{\Psi(p)}$).

- If $Z \perp T_{\Psi(p)}\Psi(\Omega)$, then $\langle Z, X\rangle_{g_E} = 0$ for any $X \in T_{\Psi(p)}\Psi(\Omega)$. In particular, since $\partial_l \Psi(p) \in T_{\Psi(p)}\Psi(\Omega)$, we have $\langle Z, \partial_l \Psi(p)\rangle_{g_E} = 0$ and, therefore,

$$\Big\langle \frac{\partial \Psi}{\partial x^l}(p), Z\Big\rangle_{g_E} g^{kl}|_p \frac{\partial \Psi}{\partial x^k} = 0.$$

- In order to verify that

$$\Big\langle \frac{\partial \Psi}{\partial x^l}(p), Z\Big\rangle_{g_E} g^{kl}|_p \frac{\partial \Psi}{\partial x^k}(p) = Z \quad \text{if} \quad Z \in T_{\Psi(p)}\Psi(\Omega),$$

it suffices to check that this is true for $Z = \partial_i \Psi(p)$, $j = 1, \dots, n$, since these vectors span $T_{\Psi(p)}\Psi(\Omega)$. Thus:

$$\Big\langle \frac{\partial \Psi}{\partial x^l}(p), \frac{\partial \Psi}{\partial x^i}(p)\Big\rangle_{g_E} g^{kl}|_p \frac{\partial \Psi}{\partial x^k}(p) = g_{il}|_p g^{kl}|_p \frac{\partial \Psi}{\partial x^k}(p) = \delta_i^k \frac{\partial \Psi}{\partial x^k}(p) = \frac{\partial \Psi}{\partial x^i}(p).$$